

EPIGRAPH OF OPERATOR FUNCTIONS

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ABSTRACT. It is known that a real function f is convex if and only if the set

$$E(f) = \{(x, y) \in \mathbb{R} \times \mathbb{R}; f(x) \leq y\},$$

the epigraph of f is a convex set in \mathbb{R}^2 . We state an extension of this result for operator convex functions and C^* -convex sets as well as operator log-convex functions and C^* -log-convex sets. Moreover, the C^* -convex hull of a Hermitian matrix has been represented in terms of its eigenvalues.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper, assume that $\mathcal{B}(\mathcal{H})$ is the C^* -algebra of all bounded linear operators on a Hilbert space \mathcal{H} . If $\dim(\mathcal{H}) = n$, then we identify $\mathcal{B}(\mathcal{H})$ with \mathbb{M}_n , the algebra of all $n \times n$ matrices. We denote by \mathbb{H}_n the set of all Hermitian matrices in \mathbb{M}_n . An operator $X \in \mathcal{B}(\mathcal{H})$ is called positive (denoted by $X \geq 0$) if $\langle Xa, a \rangle \geq 0$ for every $a \in \mathcal{H}$. If in addition X is invertible, then it is called strictly positive (denoted by $X > 0$). We denote by $\mathcal{B}(\mathcal{H})^+$ the set of all strictly positive operators on \mathcal{H} .

For a real interval J , we mean by $sp(J)$ the set of all self-adjoint operators on \mathcal{H} whose spectra are contained in J . A continuous function $f : J \rightarrow \mathbb{R}$ is said to be operator convex if $f\left(\frac{X+Y}{2}\right) \leq \frac{f(X)+f(Y)}{2}$ for all $X, Y \in sp(J)$. It is well known that [6, 7] f is operator convex if and only if the Jensen operator inequality

$$f(C^*XC) \leq C^*f(X)C \quad (1)$$

holds for every isometry C and every $X \in sp(J)$. A continuous function $f : (0, \infty) \rightarrow (0, \infty)$ is called operator log-convex [1, 8] if $f\left(\frac{X+Y}{2}\right) \leq f(X)\sharp f(Y)$ for all strictly positive operators X, Y , where the geometric mean \sharp is defined by $X\sharp Y = X^{\frac{1}{2}}\left(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\right)^{\frac{1}{2}}X^{\frac{1}{2}}$ for all strictly positive operators X and Y , see for example [6]. In the case where f is operator log-convex a sharper inequality than (1) is valid as

$$f\left(\sum_{i=1}^n C_i^* X_i C_i\right) \leq \left(\sum_{i=1}^n C_i^* f(X_i)^{-1} C_i\right)^{-1} \quad (2)$$

for all $X_1, \dots, X_n > 0$ and all $C_1, \dots, C_n \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^n C_i^* C_i = I$, see [8, Corollary 3.13].

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A set $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ is called C^* -convex if $X_1, \dots, X_n \in \mathcal{K}$ and $C_1, \dots, C_n \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^n C_i^* C_i = I$ implies that $\sum_{i=1}^n C_i^* X_i C_i \in \mathcal{K}$. This kind of convexity has been introduced by Loeb and Paulsen [9] as a non-commutative generalization of linear convexity and has been studied by many authors, see e.g. [5, 11, 12] and references therein. Typical examples of C^* -convex sets are $\{T \in \mathcal{B}(\mathcal{H}) : 0 \leq T \leq I\}$ and $\{T \in \mathcal{B}(\mathcal{H}); \|T\| \leq M\}$ for a fix scalar $M > 0$. It is evident that the C^* -convexity of a set \mathcal{K} in $\mathcal{B}(\mathcal{H})$ implies its convexity in the usual sense. For if $X, Y \in \mathcal{K}$ and $\lambda \in [0, 1]$, then with $C_1 = \sqrt{\lambda}I$ and $C_2 = \sqrt{1-\lambda}I$ we have $C_1^* C_1 + C_2^* C_2 = I$ and

$$\lambda X + (1 - \lambda)Y = C_1^* X C_1 + C_2^* Y C_2 \in \mathcal{K}.$$

But the converse is not true in general. For example if $A \geq 0$, then

$$[0, A] = \{X \in \mathcal{B}(\mathcal{H}); 0 \leq X \leq A\}$$

is convex but not C^* -convex [9]. The concept of C^* -convexity can be generalized to the sets which have a $\mathcal{B}(\mathcal{H})$ -module structure. Assume that \mathcal{M} is a $\mathcal{B}(\mathcal{H})$ -module. We say that a subset \mathcal{K} of \mathcal{M} is C^* -convex whenever $X_1, \dots, X_n \in \mathcal{K}$, $C_1, \dots, C_n \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^n C_i^* C_i = I$ implies that $\sum_{i=1}^n C_i^* X_i C_i \in \mathcal{K}$. For example, let

$$\mathcal{M} = \{(X_1, \dots, X_k); X_j \in \mathcal{B}(\mathcal{H}), j = 1, \dots, k\}.$$

Then \mathcal{M} is a $\mathcal{B}(\mathcal{H})$ -module under

$$\begin{aligned} \mathcal{M} \times \mathcal{B}(\mathcal{H}) &\rightarrow \mathcal{M} & ((X_1, \dots, X_k), T) &\mapsto (X_1 T, \dots, X_k T) \\ \mathcal{B}(\mathcal{H}) \times \mathcal{M} &\rightarrow \mathcal{M} & (S, (X_1, \dots, X_k)) &\mapsto (S X_1, \dots, S X_k). \end{aligned}$$

Now, $\mathcal{K} \subseteq \mathcal{M}$ is called C^* -convex if $X_i = (X_{i1}, \dots, X_{ik}) \in \mathcal{K}$, $C_i \in \mathcal{B}(\mathcal{H})$ ($i = 1, \dots, n$) and $\sum_{i=1}^n C_i^* C_i = I$ implies that

$$\sum_{i=1}^n C_i^* X_i C_i = \left(\sum_{i=1}^n C_i^* X_{i1} C_i, \dots, \sum_{i=1}^n C_i^* X_{ik} C_i \right) \in \mathcal{K}.$$

As an example, it is easy to see that $\mathcal{K} = \{(X_1, \dots, X_k) \in \mathcal{M}; 0 \leq X_j \leq I, j = 1, \dots, k\}$ is C^* -convex.

The epigraph of a real function f is defined to be the set

$$E(f) = \{(x, y) \in \mathbb{R} \times \mathbb{R}; f(x) \leq y\}.$$

It is known that f is a convex function if and only if $E(f)$ is a convex set in \mathbb{R}^2 . The main purpose of this paper is to present this result for operator functions. In particular, we give the connection between operator convex functions and C^* -convex sets as well as operator log-convex functions and C^* -log-convex sets. It is also shown that the C^* -convex hull of a Hermitian matrix can be represented in terms of its eigenvalues.

2. MAIN RESULT

For a continuous real function $f : J \rightarrow \mathbb{R}$, we define the operator epigraph of f by

$$\text{OE}(f) := \{(X, Y) \in \text{sp}(J) \times \mathcal{B}(\mathcal{H}); f(X) \leq Y\}.$$

The next result gives the connections between operator convex functions and C^* -convex sets.

Theorem 2.1. *A continuous function $f : J \rightarrow \mathbb{R}$ is operator convex if and only if $\text{OE}(f)$ is C^* -convex.*

Proof. Let f be operator convex. Let $(X_i, Y_i) \in \text{OE}(f)$ ($i = 1, \dots, n$) and $C_i \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^n C_i^* C_i = I$. Therefore $f(X_i) \leq Y_i$ ($i = 1, \dots, n$) and so we have by the Jensen operator inequality that

$$f\left(\sum_{i=1}^n C_i^* X_i C_i\right) \leq \sum_{i=1}^n C_i^* f(X_i) C_i \leq \sum_{i=1}^n C_i^* Y_i C_i.$$

In other words, $\sum_{i=1}^n C_i^* (X_i, Y_i) C_i \in \text{OE}(f)$ and so $\text{OE}(f)$ is C^* -convex.

Conversely, assume that $\text{OE}(f)$ is C^* -convex. For all $X_1, \dots, X_n \in \mathcal{B}(\mathcal{H})$, we have from the definition of $\text{OE}(f)$ that $(X_i, f(X_i)) \in \text{OE}(f)$. If $C_i \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^n C_i^* C_i = I$, then $\sum_{i=1}^n C_i^* (X_i, f(X_i)) C_i \in \text{OE}(f)$ by the C^* -convexity of $\text{OE}(f)$. It follows that

$$f\left(\sum_{i=1}^n C_i^* X_i C_i\right) \leq \sum_{i=1}^n C_i^* f(X_i) C_i,$$

which means that f is operator convex. \square

Let $\{f_\alpha; \alpha \in \Gamma\}$ be a family of operator convex functions and $M_\alpha \in \mathbb{R}$ for every $\alpha \in \Gamma$. By Theorem 2.1, the set $\{X \in \mathcal{B}(\mathcal{H}); f_\alpha(X) \leq M_\alpha, \forall \alpha \in \Gamma\}$ is C^* -convex. For example, consider the family f_α where $f_\alpha(t) = t^\alpha$ and $\alpha \in [1, 2]$. Then $\{X \in \mathcal{B}(\mathcal{H}); X^\alpha \leq M_\alpha, 1 \leq \alpha \leq 2\}$ is C^* -convex.

The Choi–Davis–Jensen inequality for an operator convex function $f : J \rightarrow \mathbb{R}$ asserts that $f(\Phi(X)) \leq \Phi(f(X))$ for every unital positive linear mapping Φ on $\mathcal{B}(\mathcal{H})$ and every $X \in \text{sp}(J)$, see [2, 3, 6]. Motivating by this result, we state a characterization for C^* -convex sets in \mathbb{H}_n using positive linear mappings.

Theorem 2.2. *If $\mathcal{K} \subseteq \mathbb{H}_n$, then the followings are equivalent:*

- (1) \mathcal{K} is C^* -convex;
- (2) $\sum_{i=1}^m \Phi_i(X_i) \in \mathcal{K}$ for every $X_i \in \mathcal{K}$, ($i = 1, \dots, m$) and every unital family $\{\Phi_i; i = 1, \dots, m\}$ of positive linear mappings on \mathbb{M}_n .

Proof. Assume that $\mathcal{K} \subseteq \mathbb{H}_n$ is C^* -convex. First note that if $X \in \mathcal{K}$ and if λ is an eigenvalue of X , then $\lambda I \in \mathcal{K}$. Indeed, it follows from the spectral decomposition that there exists a unitary U such that $U^* X U$ is a matrix such that λ is its k, k entry. Then $\lambda I = \sum_{i=1}^n E_{ki}^* U^* X U E_{ki} \in \mathcal{K}$, where $\{E_{ij}\}$ is the system of unit matrices. Now let $X_i \in \mathcal{K}$,

($i = 1, \dots, m$) and let $\{\Phi_i; i = 1, \dots, m\}$ be a unital family of positive linear mappings on \mathbb{M}_n . Assume that $X_i = \sum_{j=1}^n \lambda_{ij} P_{ij}$ be the spectral decomposition of X_i for $i = 1, \dots, m$ so that $\lambda_{ij} I \in \mathcal{K}$ for all i, j . Therefore

$$\sum_{i=1}^m \Phi_i(X_i) = \sum_{i=1}^m \Phi_i \left(\sum_{j=1}^n \lambda_{ij} P_{ij} \right) = \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} \Phi_i(P_{ij}) = \sum_{i=1}^m \sum_{j=1}^n C_{ij}^* \lambda_{ij} C_{ij},$$

where $C_{ij} = \sqrt{\Phi_i(P_{ij})}$. Taking into account that

$$\sum_{i=1}^m \sum_{j=1}^n C_{ij}^* C_{ij} = \sum_{i=1}^m \sum_{j=1}^n \Phi_i(P_{ij}) = \sum_{i=1}^m \Phi_i \left(\sum_{j=1}^n P_{ij} \right) = \sum_{i=1}^m \Phi_i(I) = I,$$

we get by the C^* -convexity of \mathcal{K} that $\sum_{i=1}^m \Phi_i(X_i) \in \mathcal{K}$.

Conversely, let $X_1, \dots, X_m \in \mathcal{K}$ and $C_i \in \mathbb{M}_n$ with $\sum_{i=1}^m C_i^* C_i = I$. Define positive linear mappings Φ_i on \mathbb{M}_n by $\Phi_i(X) = C_i^* X C_i$, ($i = 1, \dots, m$). Then $\{\Phi_i; i = 1, \dots, m\}$ is a unital family and so

$$\sum_{i=1}^m C_i^* X_i C_i = \sum_{i=1}^m \Phi_i(X_i) \in \mathcal{K},$$

i.e., \mathcal{K} is C^* -convex. □

The famous Choi–Davis–Jensen inequality which is a characterization of operator convex functions, can be derived from Theorem 2.1 and Theorem 2.2.

Corollary 2.3. *A continuous function $f : J \rightarrow \mathbb{R}$ is operator convex if and only if*

$$f \left(\sum_{i=1}^m \Phi_i(X_i) \right) \leq \sum_{i=1}^m \Phi_i(f(X_i)) \quad (3)$$

for every $X_i \in sp(J)$ and every unital family $\{\Phi_i\}_{i=1}^m$ of positive linear mappings on \mathbb{M}_n .

Proof. Let $f : J \rightarrow \mathbb{R}$ be operator convex. Then $\text{OE}(f)$ is C^* -convex by Theorem 2.1. For every $X_i \in sp(J)$, we have $(X_i, f(X_i)) \in \text{OE}(f)$. Theorem 2.2 then implies that $\sum_{i=1}^m \Phi_i(X_i, f(X_i)) \in \text{OE}(f)$ for every unital family $\{\Phi_i\}_{i=1}^m$ of positive linear mappings. Hence (3) holds true. Conversely, assume that (3) is valid. Let $(X_i, Y_i) \in \text{OE}(f)$, ($i = 1, \dots, m$) so that $f(X_i) \leq Y_i$. If $\{\Phi_i\}_{i=1}^m$ is a unital family of positive linear mappings, then

$$\begin{aligned} f \left(\sum_{i=1}^m \Phi_i(X_i) \right) &\leq \sum_{i=1}^m \Phi_i(f(X_i)) \quad (\text{by (3)}) \\ &\leq \sum_{i=1}^m \Phi_i(Y_i) \quad (\text{by } f(X_i) \leq Y_i). \end{aligned}$$

This concludes that $\sum_{i=1}^m \Phi_i(X_i, Y_i) \in \text{OE}(f)$. Theorem 2.2 now implies that $\text{OE}(f)$ is C^* -convex and so f is operator convex by Theorem 2.1. □

Remark 2.4. Let $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})^+$ be C^* -convex and $0 \in \mathcal{K}$. If $C \in \mathcal{B}(\mathcal{H})$ is a contraction, then $C^*XC \in \mathcal{K}$ for every $X \in \mathcal{K}$. To see this, put $D = (I - C^*C)^{\frac{1}{2}}$. Then $C^*C + D^*D = I$ and so $C^*XC = C^*XC + D^*0D \in \mathcal{K}$ for every $X \in \mathcal{K}$. It follows that if $\sum_{i=1}^n X_i \in \mathcal{K}$, then $X_i \in \mathcal{K}$ for $i = 1, \dots, n$. For if $X, Y \in \mathcal{K}$, put $C_1 = (X+Y)^{-\frac{1}{2}}X^{\frac{1}{2}}$ and $C_2 = (X+Y)^{-\frac{1}{2}}Y^{\frac{1}{2}}$ which are contractions and

$$X = C_1^*(X+Y)C_1 \quad Y = C_2^*(X+Y)C_2.$$

Definition 2.5. We say that a set $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ is C^* -log-convex if $(\sum_{i=1}^n C_i^* X_i^{-1} C_i)^{-1} \in \mathcal{K}$ for all $X_i \in \mathcal{K}$ and $C_i \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^n C_i^* C_i = I$.

If M is a positive scalar, then $\{X \in \mathcal{B}(\mathcal{H}); 0 < X \leq M\}$ is an obvious example for C^* -log-convex sets. Moreover, if $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ is C^* -log-convex, then $\mathcal{K}^{-1} = \{X^{-1}; X \in \mathcal{K}\}$ is convex in the usual sense. For if $X, Y \in \mathcal{K}^{-1}$ and $\lambda \in [0, 1]$, with $C_1 = \sqrt{\lambda}I$ and $C_2 = \sqrt{1-\lambda}I$ we have $C_1^*C_1 + C_2^*C_2 = I$ and so $(C_1^*XC_1 + C_2^*YC_2)^{-1} \in \mathcal{K}$. This follows that $\lambda X + (1-\lambda)Y = C_1^*XC_1 + C_2^*YC_2 \in \mathcal{K}^{-1}$. More generally, a set $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ is a C^* -log-convex set if and only if $\mathcal{K} \subseteq \text{Inv}(\mathcal{B}(\mathcal{H}))$ and \mathcal{K}^{-1} is a C^* -convex set, where we mean by $\text{Inv}(\mathcal{B}(\mathcal{H}))$ the set of invertible elements in $\mathcal{B}(\mathcal{H})$.

Proposition 2.6. If \mathcal{L} and \mathcal{K} are C^* -log-convex sets, then so is

$$(\mathcal{K}^{-1} + \mathcal{L}^{-1})^{-1} = \left\{ (X^{-1} + Y^{-1})^{-1}; X \in \mathcal{K}, Y \in \mathcal{L} \right\}.$$

Proof. Assume that $C_i \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^n C_i^* C_i = I$. If $Z_1, \dots, Z_n \in (\mathcal{K}^{-1} + \mathcal{L}^{-1})^{-1}$, then $Z_i = (X_i^{-1} + Y_i^{-1})^{-1}$ for some $X_i \in \mathcal{K}$ and $Y_i \in \mathcal{L}$, ($i = 1, \dots, n$). It follows from the C^* -log-convexity of \mathcal{K} and \mathcal{L} that $(\sum_{i=1}^n C_i^* X_i^{-1} C_i)^{-1} \in \mathcal{K}$ and $(\sum_{i=1}^n C_i^* Y_i^{-1} C_i)^{-1} \in \mathcal{L}$. Therefore,

$$\begin{aligned} \left(\sum_{i=1}^n C_i^* Z_i^{-1} C_i \right)^{-1} &= \left(\sum_{i=1}^n C_i^* (X_i^{-1} + Y_i^{-1}) C_i \right)^{-1} \\ &= \left(\sum_{i=1}^n C_i^* X_i^{-1} C_i + \sum_{i=1}^n C_i^* Y_i^{-1} C_i \right)^{-1} \in (\mathcal{K}^{-1} + \mathcal{L}^{-1})^{-1}, \end{aligned}$$

which implies that $(\mathcal{K}^{-1} + \mathcal{L}^{-1})^{-1}$ is C^* -log-convex. \square

Proposition 2.7. Let $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ be inverse closed in the sense that $\mathcal{K}^{-1} \subseteq \mathcal{K}$. If \mathcal{K} is C^* -log-convex, then it is C^* -convex.

Proof. Assume that \mathcal{K} is a C^* -log-convex set with $\mathcal{K}^{-1} \subseteq \mathcal{K}$. Let $C_i \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^n C_i^* C_i = I$. If $X_i \in \mathcal{K}$, then $X_i^{-1} \in \mathcal{K}$ and we have from the C^* -log-convexity of \mathcal{K} that $(\sum_{i=1}^n C_i^* X_i C_i)^{-1} \in \mathcal{K}$. It follows that $\sum_{i=1}^n C_i^* X_i C_i \in \mathcal{K}$ and so \mathcal{K} is C^* -convex. \square

The convex hull of a set \mathcal{K} in a vector space \mathcal{X} is defined to be the smallest convex set in \mathcal{X} containing \mathcal{K} . It is known that the convex hull of \mathcal{X} is the set

$$\text{CH}(\mathcal{K}) = \left\{ \sum_{i=1}^m t_i a_i; a_i \in \mathcal{K}, m \in \mathbb{N}, \sum_{i=1}^m t_i = 1 \right\}. \quad (4)$$

The C^* -convex hull [9] of a set $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ is the smallest C^* -convex set in $\mathcal{B}(\mathcal{H})$ which contains \mathcal{K} . This is the generalization of convex hull in the non-commutative setting. It is known [9, Corollary 20] that given $T \in \mathcal{B}(\mathcal{H})$, the C^* -convex hull of $\{T\}$ is the set

$$C^*-\text{CH}(T) = \left\{ \sum_i C_i^* T C_i; \sum_i C_i^* C_i = I \right\}.$$

Moreover, we define the C^* -log-convex hull of a set $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ to be the smallest C^* -log-convex set in $\mathcal{B}(\mathcal{H})$ which contains \mathcal{K} . It is easy to see that if $T \in \mathcal{B}(\mathcal{H})$ and $T > 0$, then the C^* -log-convex hull of $\{T\}$ turns out to be

$$C^*-\text{LCH}(T) = \left\{ \left(\sum_i C_i^* T^{-1} C_i \right)^{-1}; \sum_i C_i^* C_i = I \right\}.$$

When $T \in \mathbb{H}_n$, we can present the C^* -convex hull of $\{T\}$ in terms of its eigenvalues.

Theorem 2.8. *If $\lambda_1, \dots, \lambda_n$ are eigenvalues of $T \in \mathbb{H}_n$, then*

$$C^*-\text{CH}(T) = \left\{ \sum_{i=1}^n \lambda_i E_i; E_i \geq 0, i = 1, \dots, n, \sum_{i=1}^n E_i = I \right\}.$$

Proof. Let $T = \sum_{i=1}^n \lambda_i P_i$ be the spectral decomposition of T . Put

$$\Omega = \left\{ \sum_{i=1}^n \lambda_i E_i; E_i \geq 0, i = 1, \dots, n, \sum_{i=1}^n E_i = I \right\}.$$

If $X \in C^*-\text{CH}(T)$, then $X = \sum_i C_i^* T C_i$ for some $C_i \in \mathbb{M}_n$ with $\sum_i C_i^* C_i = I$. Therefore,

$$X = \sum_i C_i^* T C_i = \sum_i C_i^* \left(\sum_{j=1}^n \lambda_j P_j \right) C_i = \sum_{j=1}^n \lambda_j \sum_i C_i^* P_j C_i.$$

Putting $E_j = \sum_i C_i^* P_j C_i$, we have $E_j \geq 0$, $(j = 1, \dots, n)$ and

$$\sum_{j=1}^n E_j = \sum_{j=1}^n \sum_i C_i^* P_j C_i = \sum_i C_i^* \left(\sum_{j=1}^n P_j \right) C_i = \sum_i C_i^* C_i = I.$$

Hence, $X = \sum_{j=1}^n \lambda_j E_j$ and $\sum_{j=1}^n E_j = I$, i.e., $X \in \Omega$.

For the converse, note that the $C^*-\text{CH}(T)$ is C^* -convex and contains all eigenvalues of T . Now if $X = \sum_{j=1}^n \lambda_j E_j$ in which $\sum_{j=1}^n E_j = I$ and $E_j \geq 0$, $(j = 1, \dots, n)$, then

$$X = \sum_{j=1}^n \lambda_j E_j = \sum_{j=1}^n \sqrt{E_j} \lambda_j \sqrt{E_j} \in C^*-\text{CH}(T),$$

by C^* -convexity of $C^*-\text{CH}(T)$. □

Let $f : J \rightarrow \mathbb{R}$ be a continuous function. If $T \in \mathbb{H}_n$ has the spectral decomposition $T = \sum_{i=1}^n \lambda_i P_i$ in which the eigenvalues $\lambda_1, \dots, \lambda_n$ are contained in J , then the well

known functional calculus yields that $f(T) = \sum_{i=1}^n f(\lambda_i)P_i$. By use of Theorem 2.8, the C^* -convex hull of $f(T)$ turns to be

$$C^*\text{-CH}(f(T)) = \left\{ \sum_{i=1}^n f(\lambda_i)E_i; \quad E_i \geq 0, \quad i = 1, \dots, n, \quad \sum_{i=1}^n E_i = I \right\}.$$

The next result reveals the reason of naming C^* -log-convex sets. First note that, the notion of C^* -log-convexity can be extended to subsets of an algebra with a $\mathcal{B}(\mathcal{H})$ -module structure. For example, a set $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ is called C^* -log-convex if $(X_i, Y_i) \in \mathcal{K}$, $C_i \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^n C_i^* C_i = I$ implies that

$$\left(\sum_{i=1}^n C_i^* (X_i, Y_i)^{-1} C_i \right)^{-1} = \left(\left(\sum_{i=1}^n C_i^* X_i^{-1} C_i \right)^{-1}, \left(\sum_{i=1}^n C_i^* Y_i^{-1} C_i \right)^{-1} \right) \in \mathcal{K}.$$

Theorem 2.9. *A continuous function $f : (0, \infty) \rightarrow (0, \infty)$ is operator log-convex if and only if the set $\mathcal{K} = \{(X, Y) \in \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+; \quad f(X^{-1}) \leq Y\}$ is a C^* -log-convex set.*

Proof. Let f be operator log-convex, $C_i \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^n C_i^* C_i = I$. If $(X_i, Y_i) \in \mathcal{K}$, then $f(X_i^{-1}) \leq Y_i$. It follows from (2) that

$$f \left(\sum_{i=1}^n C_i^* X_i^{-1} C_i \right) \leq \left(\sum_{i=1}^n C_i^* f(X_i^{-1})^{-1} C_i \right)^{-1} \leq \left(\sum_{i=1}^n C_i^* Y_i^{-1} C_i \right)^{-1}.$$

Hence $\left(\left(\sum_{i=1}^n C_i^* X_i^{-1} C_i \right)^{-1}, \left(\sum_{i=1}^n C_i^* Y_i^{-1} C_i \right)^{-1} \right) \in \mathcal{K}$ and so \mathcal{K} is C^* -log-convex.

Conversely, assume that \mathcal{K} is C^* -log-convex, $C_i \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^n C_i^* C_i = I$. If $X_i \in \mathcal{B}(\mathcal{H})^+$ ($i = 1, \dots, n$), then $(X_i^{-1}, f(X_i)) \in \mathcal{K}$. Therefore

$$\left(\left(\sum_{i=1}^n C_i^* X_i C_i \right)^{-1}, \left(\sum_{i=1}^n C_i^* f(X_i)^{-1} C_i \right)^{-1} \right) \in \mathcal{K}$$

by the C^* -log-convexity of \mathcal{K} . It follows that

$$f \left(\sum_{i=1}^n C_i^* X_i C_i \right) \leq \left(\sum_{i=1}^n C_i^* f(X_i)^{-1} C_i \right)^{-1}$$

and so f is operator log-convex by (2). \square

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